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Non-Hermitian supersymmetry and singular, $\mathcal{PT}$-symmetrized oscillators

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Abstract
Hermitian supersymmetric partnership between singular potentials $V(q) = q^2 + G/q^2$ breaks down and can only be restored on certain *ad hoc* subspaces (Das A and Pernice S 1999 *Nucl. Phys.* B **561** 357). We show that within extended, $\mathcal{PT}$-symmetric quantum mechanics, the supersymmetry between singular oscillators can be completely re-established in a way which is continuous near $G = 0$ and leads to a new form of the bosonic creation and annihilation operators.

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1. Introduction

In Witten's supersymmetric quantum mechanics [1], an exceptionally important role is played by the linear harmonic oscillator (LHO)

$$H^{(\mathrm{LHO})} = p^2 + q^2.$$

A priori, one would expect that inessential modifications of $H^{(\mathrm{LHO})}$ will exhibit nice properties as well. Unfortunately, it is not so. An elementary counterexample is due to Jevicki and Rodriguez [2], who tried to combine $H^{(\mathrm{LHO})}$ with a singular harmonic oscillator (SHO) supersymmetric partner

$$H^{(\mathrm{SHO})} = p^2 + q^2 + \frac{G}{q^2} \qquad G \neq 0.$$

They discovered that the expected supersymmetric correspondence between their two models proves broken in a way attributed easily to the strongly singular spike in the potential (cf, e.g., section 12 of the review paper [3] for more details).

Recently, several authors have revisited this challenging problem [4–8]. Independently, these authors have found that a resolution of Jevicki's and Rodriguez's puzzle should be sought in a suitable regularization recipe. This partially broadened the range of Witten's supersymmetric quantum mechanics. At the same time, the ambiguity in the choice of the regularization itself remained a weak point of this promising approach. For example, in the formulation of Das and Pernice [5], 'every distinct solution' (of the given Schrödinger equation) 'corresponds to a distinct supersymmetrization'. This means that the *superpotential may cease to be state-independent*, with the partnership remaining incomplete, *projected* on a mere subspace of solutions. The subsequent re-formulations of this approach (say, in [6, 7]) weakened the latter disadvantage by further narrowing the class of the regularized superpotentials. In particular, for $0 < G < 1$, the use of the continuous superpotentials helps one to determine the spectrum via certain non-linear potential algebras (cf [7]) and/or via a suitable limiting transition to the various new and still 'solvable' delta-function-type singular barriers (cf also [8]).

All these observations encouraged and inspired our present study. We shall employ, in essence, the philosophy of our unpublished preprint [6], the key idea of which may be traced further back to the paper [9] on quartic anharmonicities in higher dimensions. There, Buslaev and Grecchi imagined that the centrifugal barrier G/q^2 is an isolated pole of an analytical potential in a complex plane. Its most natural regularization is analytical continuation mediated, say, by a *small complex shift* of the coordinate axis $q \in R \rightarrow r \in \mathbb{C}$ such that

$$r = r(x) = x - \mathrm{i}\varepsilon \qquad x \in (-\infty, \infty). \tag{1}$$

Our paper starts with a review of the properties of the eigenstates of $H^{(\mathrm{SHO})}$ in both their centrifugal and regularized interpretations (section 2). Section 3 then recollects a few basic facts about supersymmetry, amply illustrated on $H^{(\mathrm{LHO})}$ and applied, subsequently, to the regularized $H^{(\mathrm{SHO})}$. In the main part of this paper (section 4) the sets of wavefunctions are related by non-Hermitian supersymmetry (SUSY). Section 5 finally describes an interesting consequence in which a two-step application of the SUSY mapping leads to the new concept of the creation and annihilation operators.

2. Singular oscillators

2.1. Centrifugal barrier

Oscillator Hamiltonian $H^{(\mathrm{LHO})}$ is easily generalized to higher dimensions, $D = 2, 3, \ldots$. Fortunately, its partial differential Schrödinger equation

$$(-\triangle + |\vec{q}|^2)\psi(\vec{q}) = E\psi(\vec{q})$$

is superintegrable, i.e. separable in more ways [10]. In spherical coordinates with $q = |\vec{q}|$, it degenerates to the ordinary (so-called radial) differential equation

$$\begin{aligned} &H^{(\alpha)}\psi(q) = E(\alpha)\psi(q) \qquad H^{(\alpha)} = -\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{\alpha^2 - 1/4}{q^2} + q^2 \\ &\alpha = \alpha(\ell) = (D-2)/2 + \ell \qquad \ell = 0, 1, \ldots \end{aligned} \tag{2}$$

defined on the half-line at any angular momentum. This explains the exact solvability of the one-dimensional model $H^{(\mathrm{SHO})}$ since its Schrödinger equation differs from equation (2) by the shift of $G = \alpha^2 - 1/4$. In the same vein, the original isotropic harmonic oscillator and its smooth well $V \sim |\vec{q}|^2$ may be complemented by any additional singular central force $V' \sim \omega/|\vec{q}|^2$. Without any loss of separability, we redefine

$$\alpha = \alpha(\ell) = \sqrt{\omega + \left(\ell + \frac{D-2}{2}\right)^2} \qquad \ell = 0, 1, \ldots$$

and the same equation (2) can be considered.

2.2. $\mathcal{PT}$-symmetric solutions

The innocent-looking complex deformation $q \to r(x)$ of coordinates regularizes any centrifugal-like singularity $1/q^2$. This modifies, in fact, the whole quantum mechanics in the way advocated and made popular by Bender and Boettcher [11]. Their formalism works with non-Hermitian Hamiltonians which still commute with the product of parity $\mathcal{P}$ and time reversal $\mathcal{T}$. Such a type of 'weakening' of the Hermiticity can (though not necessarily) support the real spectra and specifies an extended, so-called $\mathcal{PT}$-symmetric quantum mechanics [12], intensively studied in the mathematically oriented contemporary literature [13]. The related enhanced interest in analyticity has already been proved useful in some applications, *inter alia*, in the context of perturbation theory [14], field theory [15] and, last but not least, supersymmetric quantum mechanics [16]. In principle, the ordinary Sturm–Liouville theory must be adapted to the new situation [17], the norms have to be replaced by the pseudo-norms [18], etc. All these technical aspects of $\mathcal{PT}$ symmetry may, fortunately, be skipped here as inessential since the regularization represented by equation (1) is all we shall need in what follows. In this sense, the present application of the $\mathcal{PT}$-symmetric formalism to the spiked harmonic oscillators will shift the line of coordinates and recall the $\mathcal{PT}$-symmetric analytical solution of the resulting Schrödinger equation (2) as described in [19]. For any non-negative $\alpha \geqslant 0$ which, for technical reasons, is not equal to an integer, $\alpha \neq 0, 1, 2, \ldots$, we get the spectrum

$$E = E_N^{(\varrho)} = 4N + 2\varrho + 2 \qquad \varrho = -Q \cdot \alpha \tag{3}$$

numbered by the integers $N = 0, 1, \ldots$ and the so-called quasi-parity $Q = \pm 1$. The related wavefunctions are represented in terms of the Laguerre polynomials,

$$\psi(r) = \langle r \mid N, \varrho\rangle = \frac{N!}{\Gamma(N+\varrho+1)} r^{\varrho+1/2} \exp(-r^2/2) L_N^{(\varrho)}(r^2). \tag{4}$$

The quasi-parity Q is defined in such a way that it coincides with the ordinary spatial parity P in the limit $\varepsilon \to 0$. This convention puts the quasi-even level $E^{(-\alpha)}$ with the dominating threshold behaviour $\psi(r) \sim r^{1/2-\alpha}$ lower than its quasi-odd complement $E^{(+\alpha)}$ with the dominated threshold behaviour $\psi(r) \sim r^{1/2+\alpha}$ at any fixed N. In this way, the Hermitian limit $\varepsilon \to 0$ leads to the necessity of elimination of the former, quasi-even solutions as unphysical (i.e. quadratically non-integrable) whenever $\alpha \geqslant 1$.

Our bound states degenerate to the well-known eigenstates of the linear harmonic oscillator at $\alpha = 1/2$. Marginally, let us note that in the other regularization schemes, the correspondence between P and Q may be different. The ambiguity is due to the strongly singular character of the core $1/q^2$. Thus, in the matching recipe of section 3 in [5] for example, Das and Pernice recommend an exclusive use of $Q = +1$. The spatial parity $P = \pm 1$ is then introduced in a non-analytical manner. A continuous extension of this recipe to the regular case with $\alpha = 1/2$ is, therefore, impossible.

3. Supersymmetry

For the linear harmonic oscillator, the Schrödinger factorization method [20] in application to the Hamiltonian $H^{(\mathrm{LHO})}$ offers a nice illustration of the essence of the supersymmetric quantum mechanics.

3.1. Example

Let us remind the reader that $H^{(\mathrm{LHO})} = A \cdot B - 1 = B \cdot A + 1$ with $A = q + \mathrm{i}p$ and $B = q - \mathrm{i}p$. This enables us to define a pair of partner Hamiltonians, namely, the 'left'

$H_{(L)} = H^{(\mathrm{LHO})} - 1 = B \cdot A$ and the 'right' $H_{(R)} = H^{(\mathrm{LHO})} + 1 = A \cdot B$. One can easily verify that their factorization implies that the so-called 'super-Hamiltonian' and two 'supercharges'

$$\mathcal{H} = \begin{bmatrix} H_{(L)} & 0 \\ 0 & H_{(R)} \end{bmatrix} \qquad \mathcal{Q} = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \qquad \tilde{\mathcal{Q}} = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$$

generate a representation of Lie superalgebra sl(1/1),

$$\{\mathcal{Q}, \tilde{\mathcal{Q}}\} = \mathcal{H} \qquad \{\mathcal{Q}, \mathcal{Q}\} = \{\tilde{\mathcal{Q}}, \tilde{\mathcal{Q}}\} = 0 \qquad [\mathcal{H}, \mathcal{Q}] = [\mathcal{H}, \tilde{\mathcal{Q}}] = 0.$$

In this context, the creation, annihilation and occupation-number operators are easily defined for fermions,

$$\mathcal{F}^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \mathcal{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathcal{N}_{\mathcal{F}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

In the bosonic sector, the Fock-space structure is, generically, more complicated (cf, e.g., sections 2 and 8 in the review [3]). It only becomes simplified for the present harmonic-oscillator example where the creation and/or annihilation of a boson remains mediated by the first-order differential operators $\mathbf{a}^\dagger \sim B$ and $\mathbf{a} \sim A$, respectively. This enables us to work with the factorized supercharges $\mathcal{Q} \sim \mathbf{a}\mathcal{F}^\dagger$ and $\tilde{\mathcal{Q}} \sim \mathbf{a}^\dagger\mathcal{F}$ and the following elementary vacuum:

$$\langle q \mid 0\rangle = \begin{bmatrix} \exp(-q^2/2)/\sqrt{\pi} \\ 0 \end{bmatrix} \qquad \mathcal{Q}|0\rangle = \tilde{\mathcal{Q}}|0\rangle = 0.$$

We may summarize that in this model the supersymmetry between bosons and fermions is unbroken and explicitly represented in the Fock space (cf [3], p 283).

3.2. Superpotentials

Even beyond the above elementary harmonic-oscillator illustration, all the *Hermitian* supersymmetric quantum mechanics are based on Schrödinger factorization of the Hamiltonians (cf review [3]). These constructions *start* from the so-called superpotential W and the doublet of the explicitly defined operators $A = \partial_q + W$ and $B = -\partial_q + W$. This leads us to the supersymmetry as described in section 3.1 and to the two partner Hamiltonian operators which are different from each other in general,

$$H_{(L)} = B \cdot A = \hat{p}^2 + W^2 - W' \qquad H_{(R)} = A \cdot B = \hat{p}^2 + W^2 + W'. \tag{5}$$

When we return to our regularization recipe $q \to r(x) = x - \mathrm{i}\varepsilon$, we may re-interpret our $\mathcal{PT}$-symmetric Schrödinger equation (2) as a regular complex equation on the real line of x. It is then easy to introduce the superpotential we need,

$$W^{(\gamma)}(r) = -\frac{\partial_r\langle r \mid 0, \gamma\rangle}{\langle r \mid 0, \gamma\rangle} = r - \frac{\gamma + 1/2}{r} \qquad r = r(x). \tag{6}$$

This function is regular at all real x. In other words, we start from the choice of a real parameter γ and from knowledge of the related superpotential (6) and *define* the pair (5) afterwards. In this step we already obtained an interesting pattern which is summarized in table 1. The supersymmetric recipe gives the γ-numbered partner Hamiltonians in the explicit and compact harmonic oscillator form,

$$H_{(L)}^{(\gamma)} = H^{(\alpha)} - 2\gamma - 2 \qquad H_{(R)}^{(\gamma)} = H^{(\beta)} - 2\gamma \qquad \alpha = |\gamma| \qquad \beta = |\gamma + 1|. \tag{7}$$

In light of equations (3) and (4), the energies and wavefunctions are given by the elementary formulae.

Table 1. $\mathcal{PT}$ supersymmetry of harmonic oscillators for non-integer γ.

The range of γ	$(-\infty, -1)$	$(-1, 0)$	$(0, \infty)$
Parameters			
$\alpha = \|\gamma\| > 0$	$-\gamma$	$-\gamma$	γ
$\beta = \|\gamma + 1\| > 0$	$\alpha - 1$	$1 - \alpha$	$\alpha + 1$
Hamiltonians			
$H_{(L)}$	$H^{(\alpha)} + 2\beta$	$H^{(\alpha)} - 2\beta$	$H^{(\alpha)} - 2\beta$
$H_{(R)}$	$H^{(\beta)} + 2\alpha$	$H^{(\beta)} + 2\alpha$	$H^{(\beta)} - 2\alpha$
Energies			
$E_{(R)}^{(\beta)}$	$4N + 4\alpha$	$4N + 4$	$4N + 4$
$E_{(R)}^{(\alpha)}$	$4N + 4\alpha$	$4N + 4\alpha$	$4N$
$E_{(R)}^{(-\beta)}$	$4N + 4$	$4N + 4\alpha$	$4N - 4\alpha$
$E_{(R)}^{(-\alpha)}$	$4N$	$4N$	$4N - 4\alpha$

3.3. Spectra

We have to distinguish between the three intervals of γ because the Hamiltonians depend on the absolute values $\alpha = |\gamma|$ and $\beta = |\gamma + 1|$. This implies that for the fixed parameter γ and at any principal quantum number N, we have four different wavefunctions distinguished by the subscripts $_{(L)}$ and $_{(R)}$ (or arguments α and β, respectively) and by the two quasi-parities $Q = \pm 1$. This gives the four ket vectors

$$|N, -\alpha\rangle \qquad |N, -\beta\rangle \qquad |N, +\alpha\rangle \qquad |N, +\beta\rangle$$

corresponding to the respective energies

$$E_{(L)}^{(-\alpha)} \leqslant E_{(R)}^{(-\beta)} \leqslant E_{(L)}^{(+\alpha)} \leqslant E_{(R)}^{(+\beta)}. \tag{8}$$

At any $N = 0, 1, \ldots$ these energies are ordered in a γ-independent manner. This is illustrated in figure 1 which displays the γ-dependence of the low-lying spectrum for our supersymmetrized system (7). In the figure, an interplay between the ordering and the degeneracy is made visible by the infinitesimal $\eta \to 0$ shifts of the energies,

$$L(+N) = E_{(L)}^{(-\alpha)} - 2\eta \qquad L(-N) = E_{(L)}^{(+\alpha)} + \eta \tag{9}$$

$$R(+N) = E_{(R)}^{(-\beta)} - \eta \qquad R(-N) = E_{(R)}^{(+\beta)} + 2\eta. \tag{10}$$

With all N included, all the energy levels become doubly degenerate, with the single exception of $E = 0$. This is, as we know, is characteristic of the supersymmetric quantum-mechanical models where the so-called Witten index [21] does not vanish.

We may note that the level $E = 0$ coincides with the ground-state energy $L(+0)$ if and only if $\gamma < 0$. On the opposite half-line of $\gamma > 0$, the vanishing energy $L(-0) = 0$ acquires the negative quasi-parity whereas the quasi-even and doubly degenerate ground-state energy becomes strictly negative, $L(+0) = R(+0) < 0$. The latter feature of our present consequent non-Hermitian supersymmetrization is in sharp contrast to the strict non-degeneracy of the ground states in the Hermitian cases.

4. Wavefunctions

4.1. Supercharges

The coincidence of the strengths of the spike in our two partner Hamiltonians (7) is possible but fairly exceptional. Indeed, postulating that $\alpha = \alpha_e = \beta = \beta_e$, the related parameter γ_e

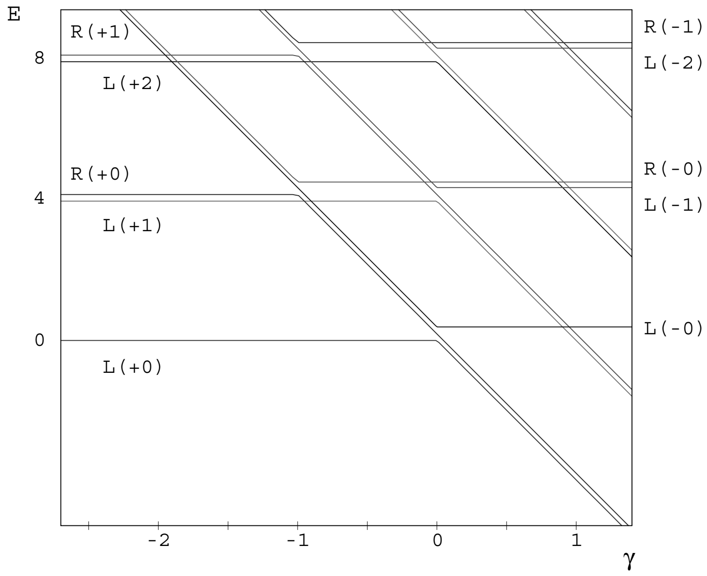

Figure 1. The γ-dependence of the SHO spectrum generated by superpotential (6).

becomes specified by the algebraic equation $|\gamma_e| = |\gamma_e + 1|$. Its solution is unique, $\gamma_e = -1/2$, and makes our superpotential (6) regular. In figure 1 we may check that such a choice gives the equidistant LHO spectrum.

All the other admissible (i.e. non-integer and real) values of γ lead to the singular supercharge components

$$A^{(\gamma)} = \partial_r + W^{(\gamma)} \qquad B^{(\gamma)} = -\partial_r + W^{(\gamma)} \qquad \gamma \neq 0, \pm 1, \ldots . \tag{11}$$

They act on our (normalized, spiked and $\mathcal{PT}$-symmetric) Laguerre-polynomial states

$$\langle r \mid N, -Q \cdot \alpha \rangle \equiv \mathcal{L}_N^{(-Q\alpha)}(r)$$

in an extremely transparent and compact manner,

$$A^{(\gamma)} \mathcal{L}_{N+1}^{(\gamma)} = c_1(N, \gamma) \mathcal{L}_N^{(\gamma+1)} \qquad c_1(N, \gamma) = -2\sqrt{N+1} \tag{12}$$

$$B^{(\gamma)} \mathcal{L}_N^{(\gamma+1)} = c_2(N, \gamma) \mathcal{L}_{N+1}^{(\gamma)} \qquad c_2(N, \gamma) = -2\sqrt{N+1} \tag{13}$$

$$A^{(\gamma)} \mathcal{L}_N^{(-\gamma)} = c_3(N, \gamma) \mathcal{L}_N^{(-\gamma-1)} \qquad c_3(N, \gamma) = 2\sqrt{N-\gamma} \tag{14}$$

$$B^{(\gamma)} \mathcal{L}_N^{(-\gamma-1)} = c_4(N, \gamma) \mathcal{L}_N^{(-\gamma)} \qquad c_4(N, \gamma) = 2\sqrt{N-\gamma}. \tag{15}$$

This is our main formula. The first two equations prove sufficient to define the well known one-dimensional annihilation and creation at $\alpha = 1/2$. The latter two find their application at any $\alpha \neq 1/2$. For the slightly non-LHO choice of $\gamma = 2/5$ this is illustrated in table 2.

We see an explicit γ-dependence of c_3 and c_4. These coefficients would vanish (and mimic a 'false-vacuum') at any integer γ. This is an additional, algebraic reason for our elimination of $\gamma =$ integer, complementing the analytical pathology of these points (namely, an unavoided level crossing) as observed previously in [19].

Table 2. The action of $A^{(\gamma)}$ near LHO, at $\gamma = -1/2 + 1/10 = -2/5$.

$E_{(L)} = E_{(R)}$	$\|N_{(L)}\rangle$	$\longrightarrow$	$\|N_{(R)}\rangle$
$\vdots$	$\vdots$		$\vdots$
8	$\mathcal{L}_2^{(-2/5)}$	$\rightarrow$	$\mathcal{L}_1^{(3/5)}$
5.6	$\mathcal{L}_1^{(2/5)}$	$\rightarrow$	$\mathcal{L}_1^{(-3/5)}$
4	$\mathcal{L}_1^{(-2/5)}$	$\rightarrow$	$\mathcal{L}_0^{(3/5)}$
1.6	$\mathcal{L}_0^{(2/5)}$	$\rightarrow$	$\mathcal{L}_0^{(-3/5)}$
0	$\mathcal{L}_0^{(-2/5)}$	$\rightarrow$	0

4.2. *Hermitian limit*

The 'natural' domain of the parameter $\gamma \notin \mathbb{Z}$ in our superpotential (6) is a real line, $\gamma \in (-\infty, \infty)$. In the Hermitian limit $\varepsilon \to 0$, this domain has to be split into five separate subdomains, namely, the 'far left' $\mathcal{D}_{(fl)} = (-\infty, -2)$, the 'near left' $\mathcal{D}_{(nl)} = (-2, -1)$, the above-mentioned 'centre' $\mathcal{D}_{(c)} = (-1, 0)$, the 'near right' $\mathcal{D}_{(nr)} = (0, 1)$ and the 'far right' $\mathcal{D}_{(fr)} = (1, \infty)$.

In the leftmost and rightmost intervals $\mathcal{D}_{(fl)}$ and $\mathcal{D}_{(fr)}$, the respective quasi-even $\mathcal{PT}$-symmetric doublets $[L(+N+1), R(+N)]$ and $[L(+N), R(+N)]$ become non-normalizable and completely disappear from our horizon. Up to that expected reduction, the limit $\varepsilon \to 0$ does not change the original $\mathcal{PT}$-symmetric spectrum. In both the latter domains, the SHO supersymmetry is established in a more or less textbook form.

Within the neighbouring two additional intervals $\mathcal{D}_{(nl)}$ and $\mathcal{D}_{(nr)}$, the SUSY would be completely destroyed because of the survival of the respective quasi-even normalizable solutions $\mathcal{L}_N^{(-\beta)}(r)$ or $\mathcal{L}_N^{(-\alpha)}(r)$. Similar 'redundant' sets of solutions have already been reported as causing serious difficulties [2, 3, 5]. In light of our present results, the supersymmetic correspondence between the Hamiltonians $H^{(\alpha)}$ and $H^{(\beta)}$ may again be fully restored within the latter two domains. One need not even resort to any sophisticated reasons because the necessary elimination of the SUSY-breaking wavefunctions can simply be performed by using an auxiliary boundary condition at the origin,

$$\lim_{r \to 0} \frac{\psi(r)}{\sqrt{r}} = 0. \tag{16}$$

Fortunately, this condition coincides with the standard physical constraint for the radial wavefunctions in higher dimensions [22]. Hence, we may easily interpret this constraint as a mere return to the standard SUSY *without* singularities. Indeed, our superpotential W has no singularities within the range $r \in (0, \infty)$ of the radial coordinate at $D > 1$.

A remarkable situation is encountered in $\mathcal{D}_{(fl)}$ and $\mathcal{D}_{(nl)}$ where our $\varepsilon \to 0$ supersymmetry could be characterized, conventionally, as broken (cf p 285 in [3]). Its higher dimensional re-interpretation in $\mathcal{D}_{(nl)}$ becomes necessary again.

One of our most amazing conclusions concerns the 'central' interval $\mathcal{D}_{(c)}$ where our $\mathcal{PT}$-symmetric regularization can be very easily removed and the picture provided by figure 1 applies in the Hermitian case without any changes.

We may conclude that the $\mathcal{PT}$-symmetric formalism leads to the Hermitian limits which exhibit the correct supersymmetric correspondence between Hamiltonians (7) for almost all the parameters γ. During the limiting transition $\varepsilon \to 0$, the Hermitian spectra may be reduced but the supersymmetry survives. Roughly speaking, we re-established a full supersymmetry between the 'left' and 'right' SHO systems simply via their s-wave re-interpretation. This conclusion is summarized in table 3.

Table 3. Hermitian limit $\varepsilon \to 0$ in figure 1 and supersymmetric correspondence between the spiked harmonic oscillators (7).

Domain	(fl)	(nl)	(c)	(nr)	(fr)
γ	$(-\infty, -2)$	$(-2, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$\triangle$[a]	0[b]	0[b]	1[c]	1[d]	1[d]
$E_{(L)}$[e]	$L\,(-N)$	$L\,(-N)$	$L\,(+N+1)$[f]	$L\,(-N-1)$	$L\,(-N-1)$
$\mathcal{L}_N^{(-\alpha)}(r)$[g]	Absent[h]	Absent[h]	Present	Dropped[i]	Absent[h]
$\mathcal{L}_N^{(-\beta)}(r)$[j]	Absent[h]	Dropped[i]	Present	Absent[h]	Absent[h]
SUSY	Broken	Broken	Unbroken	Unbroken	Unbroken

[a] Witten's index [21].
[b] Degenerate ground state at positive energy $L\,(-0) = R\,(-0) = 4\alpha$.
[c] Non-degenerate ground state at energy $L\,(+0) = 0$.
[d] Non-degenerate ground state at energy $L\,(-0) = 0$.
[e] Supersymmetric partner of $E_{(R)} = R\,(-N)$.
[f] The second series has $E'_{(L)} = E'_{(R)} = L(-N) = R(+N)$.
[g] Quasi-even state with energy $L\,(+N)$.
[h] Not integrable.
[i] Eliminated using an auxiliary boundary condition at the origin.
[j] Quasi-even state with energy $R\,(+N)$.

5. Innovated annihilation and creation

5.1. Definition

At $\gamma = -1/2$ we encounter the 'degenerate' (and, in the present context, utterly exceptional) textbook LHO pattern

$$A^{(-1/2)}\mathcal{L}_{N-1}^{(1/2)}(q) \sim \mathcal{L}_{N-1}^{(-1/2)}(q) \qquad A^{(-1/2)}\mathcal{L}_{N}^{(-1/2)}(q) = -\sqrt{2N}\mathcal{L}_{N-1}^{(1/2)}(q)$$
$$B^{(-1/2)}\mathcal{L}_{N-1}^{(-1/2)}(q) \sim \mathcal{L}_{N-1}^{(1/2)}(q) \qquad B^{(-1/2)}\mathcal{L}_{N-1}^{(1/2)}(q) \sim \mathcal{L}_{N}^{(-1/2)}(q).$$

The second half of table 4 (which, as a whole, will be needed later) offers a remarkable alternative. Indeed, via the non-Hermitian detour and limit $\varepsilon \to 0$, another explicit annihilation pattern is obtained for the same s-wave oscillator. The new SUSY mapping would start from the Hamiltonian $H_{(L)} = H^{(1/2)} - 3$ giving its $\mathcal{PT}$-symmetrically regularized non-Hermitian partner $H_{(R)} = H^{(3/2)} - 1$. In the subsequent step (and in a way indicated, up to the shifts which are different, in the first half of table 4), the similar SUSY partnership of the re-shifted $H_{(L)} = H^{(3/2)} + 1$ would return us to the re-shifted original $H_{(R)} = H^{(1/2)} + 3$.

All these examples indicate that the annihilation operators and their creation partners can be introduced in the factorized, second-order differential form

$$A^{(-\gamma-1)} \cdot A^{(\gamma)} = A^{(\gamma-1)} \cdot A^{(-\gamma)} = \mathbf{A}(\alpha) \tag{17}$$

$$B^{(-\gamma)} \cdot B^{(\gamma-1)} = B^{(\gamma)} \cdot B^{(-\gamma-1)} = \mathbf{A}^{\dagger}(\alpha). \tag{18}$$

Once we start from $\alpha = 3/2$ this observation may be illustrated by the two alternative superpositions of the action of the supercharges $A^{(\gamma)}$ as displayed in tables 4 and 5. In the former case the $\gamma = -3/2$ $\mathcal{PT}$-supersymmetry between $H_{(L)} = H^{(3/2)} + 1$ and $H_{(R)} = H^{(1/2)} + 3$ is followed by the $\gamma = 1/2$ correspondence between the doublet $H_{(\tilde{L})} = H^{(1/2)} - 3$ and $H_{(\tilde{R})} = H^{(3/2)} - 1$. As a net result, we obtain the appropriate generalization of the annihilation pattern for the harmonic oscillator in p-wave. Table 5 offers an alternative path.

Table 4. Singular Hamiltonian $H^{(3/2)} = p^2 + (x - i\varepsilon)^2 + 2/(x - i\varepsilon)^2$ and an annihilation operator as a double supersymmetric mapping with initial $\gamma = -3/2$.

$E_{(L)} = E_{(R)}$	$\|N_{(L)}\rangle$	$\xrightarrow{A^{(-3/2)}}$	$\|N_{(R)}\rangle = \|N_{(\tilde{L})}\rangle$	$\xrightarrow{A^{(1/2)}}$	$\|N_{(\tilde{R})}\rangle$	$E_{(\tilde{L})} = E_{(\tilde{R})}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
14	$\mathcal{L}_2^{(3/2)}$	$\rightarrow$	$\mathcal{L}_2^{(1/2)}$	$\rightarrow$	$\mathcal{L}_1^{(3/2)}$	8
12	$\mathcal{L}_3^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_2^{(-1/2)}$	$\rightarrow$	$\mathcal{L}_2^{(-3/2)}$	6
10	$\mathcal{L}_1^{(3/2)}$	$\rightarrow$	$\mathcal{L}_1^{(1/2)}$	$\rightarrow$	$\mathcal{L}_0^{(3/2)}$	4
8	$\mathcal{L}_2^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_1^{(-1/2)}$	$\rightarrow$	$\mathcal{L}_1^{(-3/2)}$	2
6	$\mathcal{L}_0^{(3/2)}$	$\rightarrow$	$\mathcal{L}_0^{(1/2)}$	$\rightarrow$	0	0
4	$\mathcal{L}_1^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_0^{(-1/2)}$	$\rightarrow$	$\mathcal{L}_0^{(-3/2)}$	−2
2	——		——		——	−4
0	$\mathcal{L}_0^{(-3/2)}$	$\rightarrow$	0	$\rightarrow$	——	−6

Table 5. Same as table 4 with $\gamma = +3/2$.

$E_{(L)} = E_{(R)}$	$\|N_{(L)}\rangle$	$\xrightarrow{A^{(3/2)}}$	$\|N_{(R)}\rangle = \|N_{(\tilde{L})}\rangle$	$\xrightarrow{A^{(-5/2)}}$	$\|N_{(\tilde{R})}\rangle$	$E_{(\tilde{L})} = E_{(\tilde{R})}$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
8	$\mathcal{L}_2^{(3/2)}$	$\rightarrow$	$\mathcal{L}_1^{(5/2)}$	$\rightarrow$	$\mathcal{L}_1^{(3/2)}$	14
6	$\mathcal{L}_3^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_3^{(-5/2)}$	$\rightarrow$	$\mathcal{L}_2^{(-3/2)}$	12
4	$\mathcal{L}_1^{(3/2)}$	$\rightarrow$	$\mathcal{L}_0^{(5/2)}$	$\rightarrow$	$\mathcal{L}_0^{(3/2)}$	10
2	$\mathcal{L}_2^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_2^{(-5/2)}$	$\rightarrow$	$\mathcal{L}_1^{(-3/2)}$	8
0	$\mathcal{L}_0^{(3/2)}$	$\rightarrow$	0	$\rightarrow$	——	6
−2	$\mathcal{L}_1^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_1^{(-5/2)}$	$\rightarrow$	$\mathcal{L}_0^{(-3/2)}$	4
−4	——		——		——	2
−6	$\mathcal{L}_0^{(-3/2)}$	$\rightarrow$	$\mathcal{L}_0^{(-5/2)}$	$\rightarrow$	0	0

5.2. Action

At a general $\alpha \neq 0, 1, 2, \ldots$, the operators (17) and (18) enable us to move along the spectrum of any spiked harmonic oscillator Hamiltonian $H^{(\alpha)}$. We get the elementary and transparent action on all the solutions,

$$\mathbf{A}(\alpha) \cdot \mathcal{L}_{N+1}^{(\gamma)} = c_5(N,\gamma)\mathcal{L}_N^{(\gamma)}$$

$$\mathbf{A}^\dagger(\alpha) \cdot \mathcal{L}_N^{(\gamma)} = c_5(N,\gamma)\mathcal{L}_{N+1}^{(\gamma)}$$

$$c_5(N,\gamma) = -4\sqrt{(N+1)(N+\gamma+1)} \qquad \gamma = \pm\alpha.$$

We achieved a unified description of the spiked harmonic oscillators $H^{(\alpha)}$ within the $\mathcal{PT}$-symmetric framework:

- The $\mathcal{PT}$-supersymmetric partnership is mediated by the first-order differential operators $A^{(\gamma)}$ and $B^{(\gamma)}$.
- For any non-integer $\alpha > 0$ in the Hamiltonian $H^{(\alpha)}$, the role of the creation and annihilation operators is played by the α-dependent and γ-preserving differential operators $\mathbf{A}^\dagger(\alpha)$ and $\mathbf{A}(\alpha)$ of second order.

The $\mathcal{PT}$-supersymmetric partners coincide only in the regular case. Their traditional creation and annihilation operators $\mathbf{a}^\dagger \sim B(-1/2)$ and $\mathbf{a} \sim A(-1/2)$ *change* the quasi-parity. This feature is not transferable to any non-equidistant spectrum with $\gamma \neq -1/2$.

Our 'natural' operators of creation $\mathbf{A}^\dagger(\alpha)$ and annihilation $\mathbf{A}(\alpha)$ are smooth near $\alpha = 1/2$. Their marginal (though practically relevant) merit lies in their reducibility to their regular first-order differential representation

$$\mathbf{A}(\alpha) \cdot \mathcal{L}_N^{(\gamma)} = \left(2r\partial_r + 2r^2 - 4N - 2\gamma - 1\right) \mathcal{L}_N^{(\gamma)}$$
$$\mathbf{A}^\dagger(\alpha) \cdot \mathcal{L}_N^{(\gamma)} = \left(-2r\partial_r + 2r^2 - 4N - 2\gamma - 3\right) \mathcal{L}_N^{(\gamma)}$$

which is, of course, state-dependent. A further change of variables $r \to y$ such that $r = \exp 2y$ gives a simpler differentiation $2r\partial_r \to \partial_y$ and the Morse Hamiltonians with $\mathcal{PT}$-symmetry [23]. This indicates that the Morse potentials would also deserve more attention in the supersymmetric context.

6. Summary

In their inspiring paper [2], Jevicki and Rodriguez emphasized that the supersymmetric partnership cannot be postulated between $H_{(L)} = H^{(\mathrm{LHO})} - 3$ (with energies $E_0^{(+1/2)} = -2$, $E_0^{(-1/2)} = 0$, $E_1^{(+1/2)} = 2$, $E_1^{(-1/2)} = 4$, etc) and $H_{(R)} = p^2 + q^2 + 2/q^2 - 1$ (with a different set of the levels $E_0^{(-3/2)} = 4$, $E_1^{(-3/2)} = 8$, etc). We have seen that the puzzle is resolved when we treat both operators as s-wave Hamiltonians. This reduces the 'left' spectrum to the new set ($E_0^{(-1/2)} = 0$, $E_1^{(-1/2)} = 4$, etc) and the SUSY is restored.

The problem recurred when Das and Pernice [5] did not find any analogy between $\alpha = 1/2$ (smooth, LHO) and $\alpha \neq 1/2$ (spiked, singular, SHO). In their method, different approaches were required as long as a few *a priori* SUSY-supporting requirements (e.g. the existence of a non-degenerate ground state at $E = 0$) were postulated. As a consequence, the supersymmetry of [5] did not apply to the pairs of operators but rather to their *ad hoc* projections which were not always clearly specified.

The non-analytical regularization of [5] was also unnecessarily complicated. For example, the regularization giving the even quasi-parity $Q = -1$ (as mentioned at the end of section 2.2) was only chosen consequently at the even spatial parity $P = -1$. For $P = +1$ one used $Q = -1$ at $\gamma > 0$ for the 'left' $H_{(L)}$ and at $\gamma < -1$ for the 'right' $H_{(R)}$. For the other γ it was necessary to use the quasi-even solutions with $Q = +1$, anyhow.

These problems have been resolved in the present alternative approach. We have shown that a key to the problem lies in the suitable non-Hermitian regularization of the singular superpotentials. Although this merely circumvents the problem with the singularity in $H^{(\mathrm{SHO})}$, we need not really remove the regularization in the majority of phenomenological and supersymmetric applications. It suffices, mostly, to stay suitably (though not too much) close to the limit, keeping the Schrödinger equations comfortably non-singular. Moreover, there exist serious mathematical reasons why one should avoid the removal of the regularization whenever possible. In one dimension, the $1/q^2$ barrier *always* separates the real line, strictly speaking, into two non-communicating halves [24].

In our paper we have advocated the use of the $\mathcal{PT}$-symmetric regularization (1) which exhibits several specific merits. First of all, it 'supersymmetrizes' the pairs of Hamiltonians $H^{(\mathrm{SHO})}$ for all the couplings $G = \alpha^2 - 1/4$ for which $\alpha = |\gamma|$ is not an integer, $\gamma \notin \mathbb{Z}$. Secondly, all the formulae degenerate to the well-known harmonic oscillator SUSY at $\gamma = -1/2$. Thirdly, the limiting transition $\varepsilon \to 0$ proves smooth at all the neighbouring $\gamma \in (-1, 0)$.

This enabled us to generalize the SUSY LHO model to all the SHO doublets $H_{(L)}^{(\alpha)}$ and $H_{(R)}^{(\beta)}$ with $\alpha = |\gamma|$ and $\beta = |\gamma + 1|$.

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